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Theoretical-graph models of large-scale clustering of matter in the Universe

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Abstract. The paper reports on construction of a theoretical-graph model of large-scale clustering of matter in the Universe. This model allows for the expected fractal structure of the Universe. A topological characteristic of the pre-fractal graph, for which there exists astrophysical interpretation, is obtained.

Key words: cosmology: large-scale structure of the Universe — galaxies: clusters — cosmology: models

A firm point of view on the fractal structure of the Universe has been established by the present time. The morphological description of clustering in the Universe reduces, as noted by Vavilova (1995) and Lebedev (1988), to the following: most galaxies are members of groups and clusters. Groups consist, on the average, of a few dozen galaxies and form clusters (Baryshev, 1981; Lebedev, 1988; Lukkin, 1988) whose maximum diameter is about 10 Mpc.

Statistical data acquired so far allow one to speak of the existence of superclusters (Baryshev, 1981; Lebedev, 1988; Lukkin, 1988), or, in other words, second-level (second structural level) clusters. Superclusters comprise several clusters and are about 10^2 Mpc in size.

The Methagalaxy, i.e. the visible Universe, the diameter of which is about 6500 Mpc, is considered as a third-level cluster. From the available morphological description it follows that stellar systems are structurally co-ordinative, i.e. the hierarchical structure of the Universe does exist (Baryshev, 1981).

The distribution of galaxies has concentrations and rarefactions on all scales. In numerous publications the latter was the basis for the quantitative conclusion on the fractal spatial distribution of matter in the Universe. One should pay attention to the fact that the fractal dimension is different for different levels. As it is pointed out by Feitzinger and Galinski (1987), for spiral galaxies $d \approx 1.68$. According to Feitzinger and Galinski (1987); Einasto (1988); Vicsek and Szaley (1987), the fractal dimension of a galaxy cluster is estimated as $d \approx 1.2$. As to the fractal dimension of superclusters, different authors give different values, from $d \approx 1.3$ (Klypin et al., 1989) to $d \approx 2$ (Lebedev, 1988; Wen et al., 1989). The fractality of the large-scale structure of the Universe is also evidenced by the self-similarity of voids (Einasto et al., 1989).

In the present paper we made an attempt to abstract from various values of fractal dimension for different levels and based on a common approach to construct a hierarchical mathematical model which would represent the fractal structure of the Universe.

The proposed mathematical model of large-scale clustering of matter in the Universe (hereafter model) is based on the mathematical apparatus of fractal graph theory (Kochkarov et al., 1996; Kochkarov and Perepelitsa, 1996; 1997). Note that the graph is a pair (V, E), where $V = \{v\}$ is a set of vertices, and E is a set of edges, where the edge $e \in E$ is a pair $e = (v', v'') \in V \times V$, which is graphically presented by a portion of a curve or a straight line in space.

The graph is a structure consisting of vertices and edges. Each vertex denotes some separate object or a part of it. The edge shows connection between separate objects. This connection may have a direction and a certain "length". Thus, a mathematical model of a discrete system, which is a graph, formalizes strongly the initial task, i.e. in the obtained model we use only the terms "vertex" (node) and "edge"(connection). The lacking definitions of concepts of the theory of graphs can be found in (Emelichev et al., 1990; Kharari, 1973).

The term fractal (pre-fractal) is associated with the infinite (finite) graph. Define these terms by way of describing the very process of graph generation. Begin the description of graph generation with the simpliest case when a single n-vertex connected graph $H = (W, Q)$ with a set of vertices W and a set of edges Q is specified as the primer (Feder, 1991). The process of pre-fractal graph generation $G = (V, E)$ is realized stepwise, each step $r = 2, 3, ..., L$ results in the current pre-fractal graph $G_r = (V_r, E_r)$ of rank r. Graph G_r is derived by means of application to the preceding graph $G_{r-1} = (V_{r-1}, E_{r-1})$ of the replacement operation of the vertex by the primer (RVP). The operation RVP is applied to each vertex $v \in V_{r-1}$ and is the generalization of the known operation of "graph vertex splitting" (Emelichev et al., 1990). The essence of this generalization is that each split vertex $v \in V_{r-1}$ is replaced not by the edge (Emelichev et al., 1990) but by the primer $H = (W, Q)$. This operation being performed, all the edges $e \in E_{r-1}$ are preserved and called old edges with respect to all current graphs $G_r, G_{r+1}, ..., G_L$. In so doing all old edges, incident to vertex $v \in V_{r-1}$ being replaced, become occasionally or regularly incident to some vertices of the primer that has replaced vertex v. The edges of each of the appearing primers are called new edges, i.e. the set of new edges is the set $(E_r \_ E_{r-1})$; the edges of this set are old in the current graph G_{r+1}.

The process of generation of graph G_r is called a stage. Number the stages by the index $r = 1, 2, ..., L$, where r is the number of the stage at which graph $G_L = G$ is obtained. The given primer H is always adopted as G_1. The term "trajectory" is used to call the sequence

$$G_1, G_2, ..., G_L, \quad (1)$$

consisting of current graphs G_r each of which is the result of operation of stage $r \in \{1, 2, ..., L\}$.

Considering pre-fractal graph $G = G_L = (V_L, E_L)$, divide the set of its edges $E = E_L$ into subsets (E_r / E_{r-1}) of the edges of rank $r, r = \overline{1, L}$, here $E_0 = \emptyset$. The generalization of the above described process of generation of pre-fractal graph G is the case when a set of primers $\mathbf{H} = \{H\} = \{H_1, H_2, ..., H_t, ..., H_T\}, T \geq 2$ is specified instead of a single primer H. The idea of this generalization is that when passing from graph G_{r-1} to graph G_r, each vertex is replaced by primer $H_t \in \mathbf{H}$, which is selected either randomly or according to a definite rule reflecting the character of the task or process being modelled.

The distinguishing feature of the above processes of generation of pre-fractal graph is that at each stage $r = 2, 3, ..., R$ in the current graph $G_{r-1} = (V_{r-1}, E_{r-1})$, each vertex $v \in V_{r-1}$ is replaced by some primer. We call the pre-fractal graph obtained as a result of such a process canonic.

In a general case the pre-fractal graph is initiated by a set of primers $\mathbf{H}$ of power $| \mathbf{H} | \geq 1$, with one principal distinction, however: in passing from graph G_{r-1} to G_r, not every vertex $v \in V_{r-1}$ is replaced by the primer from $\mathbf{H}$, but only the subset of vertices $V^*_{r-1} \subset V_{r-1}$, which is determined either at random or by some particular rule representing the specific character of the task or process being simulated.

In the geometry of fractals there is the concept of likeness coefficient (Feder, 1991), or in other terms scaling coefficient (Shuster, 1996; Kurdyumov et al., 1996). In fractal graphs the scaling coefficient is determined as the ratio of the edge length of rank $r+1$ to that of rank r. However, from the point of view of real situations, this specified scaling value may be spoken about provided that an extremely idealized model is constructed.

A more adequate model representation of a real simulated situation can be achieved if the scaling coefficient is taken from a definite interval in accordance with the processes observed. An alternative, possibly even more adequate approach, is that the edge lengths of each rank are selected in an appropriate interval, and the intervals referring to different ranks do not intersect.

The pre-fractal graph underlying the mathematical model of large-scale clustering of matter in the Universe, presumed in the paper, is derived in the following way. The set of primers $\mathbf{H}$ consists of n_t-vertex primers-graph, $t = 1, 2, ..., T$. We will recall that a graph-primer is a complete two-fraction graph, in which one of the fractions consists of a single vertex. Parameters n_t are determined by statistical (possibly hypothetical) data on the number of galaxies in clusters, clusters in superclusters, etc. In particular, a 2-vertex primer, i.e. an edge, may be the element of the set $\mathbf{H}$. Let L be the rank of the modelled Methagalaxy. A model, supposed for it, is constructed as trajectory (1) in which G_1 is a definite primer $H_1 \in \mathbf{H}$. The primer H_1 is an n_1-vertex primer, where n_1 is the number of superclusters, that is clusters, the level of which precedes the level of the modelled Methagalaxy, in other words, n_1 comprises all the galaxies, including isolated ones, that are not members of any cluster of a higher level, but for the Methagalaxy being considered. The gravitational centre of this Methagalaxy is assigned to be definitely corresponding to the centre of graph-primer H_1. The above mentioned superclusters are assigned to be definitely corresponding to pendent vertices $v_1, v_2, ..., v_i, ..., v_{n_0}; n_0 = n_1 - 1$ of primer H_1. From the view point of morphological description of the Universe the term "Methagalaxy primer" is more convenient for primer H_1. Each edge of this primer is assigned a length whose value is chosen from the interval $[a_1, b_1]$, where the quantity $a_1 + b_1$ equals half the Methagalaxy diameter, and the length of this interval is no longer than the radius of the Methagalaxy. As to astrophysical sense of the model, each pendent vertex of primer H_1 corresponds definitely to the gravitational centre of a definite supercluster. From the formal mathematical point of view, if superclusters of matter are represented by one galaxy, the corresponding vertex of the primer is then blocked, that is this vertex is not replaced by the primer when constructing current graphs of the next ranks of trajectory (1).

Current graph G_2 in trajectory (1) is determined in the following manner. Each pendent vertex of graph $G_1 = H_1$ is replaced by a definite primer, which has sense "primer of superclusters", denote the subset of all such primers of superclusters from $\mathbf{H}$ by $\mathbf{H_2}$ implying here that $\mathbf{H_1}$ consists of a single element of primer H_1 of the Methagalaxy. The lengths from the interval $[a_2, b_2]$, where a_2+b_2 is equal to half the diameter of a supercluster, and the length of this interval does not exceed the supercluster radius, are assigned to the edges of primer $\mathbf{H_2}$. Each pendent vertex of any primer selected from $\mathbf{H_2}$ corresponds definitely to a particular cluster incorporated in the supercluster. Here each old edge is incident, on the one hand, to the H_1 primer centre and, on the other hand, to the primer which replaced the pendent vertex of this edge. Note that the ratio of the edge lengths of the Methagalaxy primer to the edge lengths of primers of superclusters has an order of magnitude of the scaling index $\eta \approx 50$. From the formal point of view, as in the previous case, clusters may be represented by a single galaxy. In this case the appropriate vertex of the primer is blocked, that is this vertex is not replaced by the primer when constructing current graphs of the next ranks of trajectory (1).

Current graph G_3 in trajectory (1) is described as follows. Every vertex of graph G_2, but for the blocked ones, is replaced by a definite primer, which has sense "primer of the cluster", denote the subset of all these primers from $\mathbf{H}$ by $\mathbf{H_3}$. The same as in the previous case, the edges of primer $\mathbf{H_3}$ are assigned the lengths from the interval $[a_3, b_3]$, where the value of $a_3 + b_3$ equals half the cluster's diameter, and the length of the interval does not exceed the cluster's radius. Each of the pendent vertices of any primer selected from $\mathbf{H_3}$ is definitely related to a certain galaxy, member of the cluster. In this case, each old edge is incident to the centre of the primer from $\mathbf{H_3}$ and, alternatively, to the centre of the primer that has replaced the pendent vertex of this edge. Note that similar to preceding graph G_2, the ratio of the edge lengths of the primers of superclusters to those of clusters has an order of magnitude of the scaling coefficient $\eta \approx 50$.

The theoretical graph model of the Universe described above is characterized by the following parameters: L is the rank of the model fractal graph which represents $(L+1)$ hierarchical levels-formations (the first level formation — Methagalaxy, the second — superclusters, the third — clusters, the fourth — galaxies; n_t is the number of vertices in a primer of type t; η is the scaling coefficient to estimate the ratio of edge lengths of neighbouring ranks; edge lengths of rank r take values in the interval $[a_r, b_r]$; $[\alpha_k, \beta_k]$ is the interval of values of diameters D_k of hierarchical formations of the k-level, $k = \overline{1,4}$.

To formulate some conclusions from the above fractal model consideration, note the validity of some relations between the aforementioned parameters. With allowance made for the gravitational laws it can be stated that the distance between a couple of structures of the same level is at least an order of magnitude over the maximum diameter of these structures, i.e.

$$(b_k - a_k) > 10(\beta_k - \alpha_k), k = \overline{1,3}. \qquad (2)$$

The latter implies that the minimum separation between clusters at least an order of magnitude exceeds the maximum diameter of a cluster, the minimum separation between superclusters at least an order of magnitude exceeds the maximum diameter of a supercluster (Lebedev, 1988).

From the said above one can analyze the point of existence or absence of other universes from the point of view of their observability. If any other universe exists along with ours, one can speak about the pre-fractal graph of rank $L = 4$, or, that is the same, about five hierarchical levels of formations: galaxies, clusters, superclusters, Methagalaxies, and superuniverse. These formations are related to the following value intervals of their diameters: $[\alpha_4, \beta_4]$, $[\alpha_3, \beta_3]$, $[\alpha_2, \beta_2]$, $[\alpha_1, \beta_1]$. Then, assuming that for rank $r = 1$ the former scaling factor, $\eta \approx 50$, is valid, then in view of (2) the distance from our Universe to the nearest Methagalaxy is at least an order of magnitude over our Universe diameter, i.e. the distance ρ from us to the nearest universe satisfies the inequality

$$\rho > 10\beta_1 \approx 10 \cdot 6500\text{Mpc} = 65 \cdot 10^3\text{Mpc}. \qquad (3)$$

It follows from (3) that it must take 195 billion years for light to cover the distance from the nearest Methagalaxy to our Universe, i.e. this time exceeds the age of our Galaxy nearly 10 times.

Let us dwell on one more inference which can in principle be formulated in terms of the theoretical graph model. If the topological distance to a couple of objects is different, their velocities will then represent values of different orders, the lower velocity being that of the object to which the topological distance is smaller. As topological distance between the vertices name the number of edges in the chain that links the pair of vertices of the fractal graph.

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