Lagrangian ring

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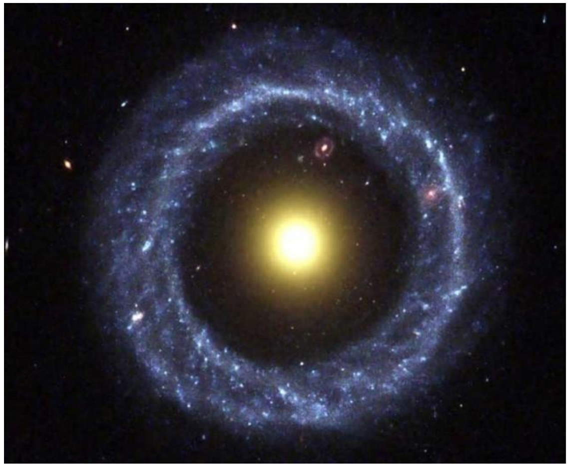
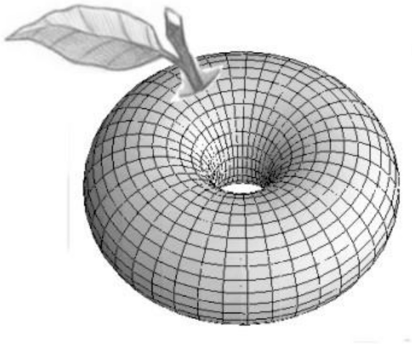
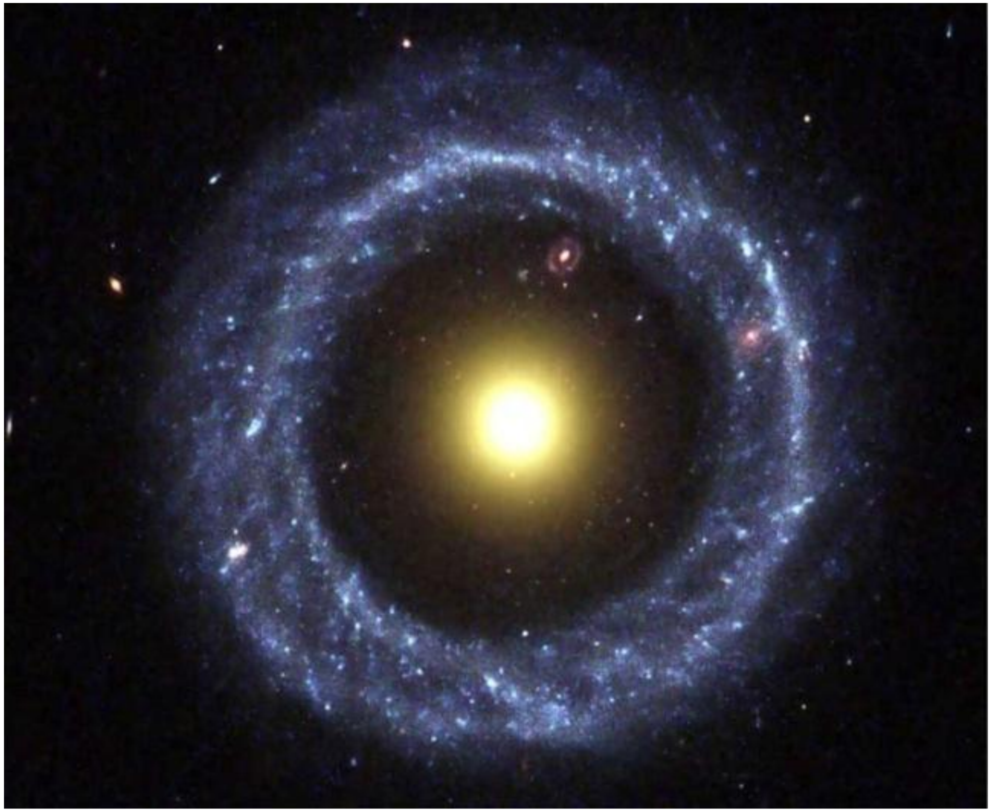




What is gravitational properties of torus?

- Ring is a particular case of the torus



Plan

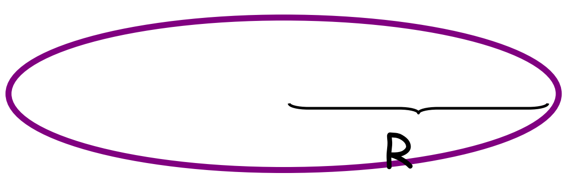
- Gravitational potential of a torus (ring)
  - "Newton's theorem" for torus
  - Role of the central mass for stability of a self-gravitating torus
- Particle trajectories in the outer potential of a torus and a central mass
  - Lagrangian ring
  - Region of unstable orbits
- Self-gravitating torus in the field of a central mass: N-body simulation
  - Keplerian torus
  - Equilibrium cross-section of self-gravitating torus
- Application to ring galaxies(?)
- What can you see if a gravitational lens is a ring galaxy?

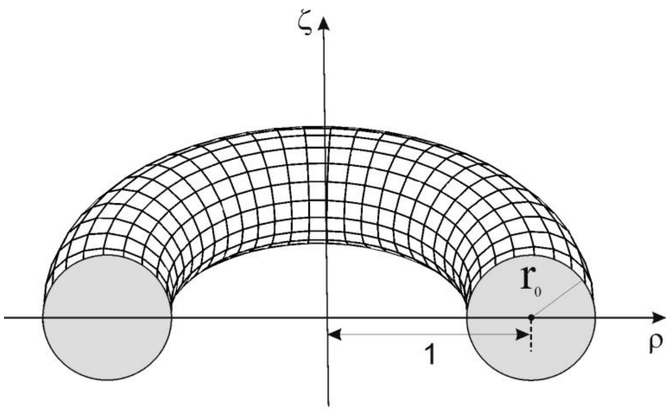
Gravitational potential of a torus

B. Riemann devoted one of his last works to
the gravitational potential of a homogeneous torus
"About Potential of a Torus" (1864)

Our idea:

We compose a torus of a set of infinitely thin rings .
Potential of a torus is a sum of potentials of such rings.

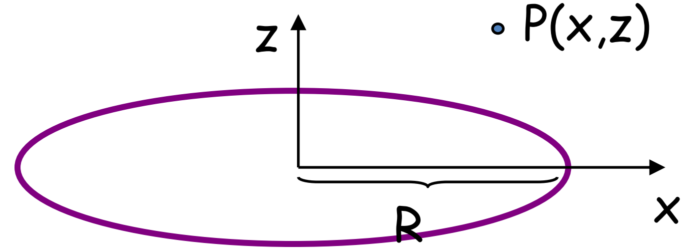




Grav. potential of a homogeneous circular torus

Grav. potential of infinitely thin ring
(a material circle)

$$\boxed{\varphi_{ring}(x,z) = \frac{GM_{ring}}{\pi R}\sqrt{\frac{m}{\rho}}\cdot K(m) = \frac{GM_{ring}}{\pi R}\phi_{ring}}$$



$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\cdot\sin^2\theta}}$$ is the complete elliptical integral of the first kind

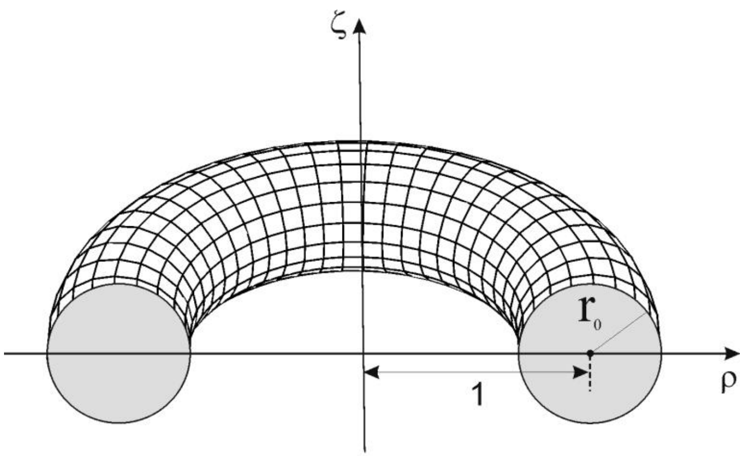
$$m = \frac{4\rho}{(\rho+1)^2 + \zeta^2}$$ is the parameter of elliptical integral

$$\rho = x/R; \quad \zeta = z/R$$

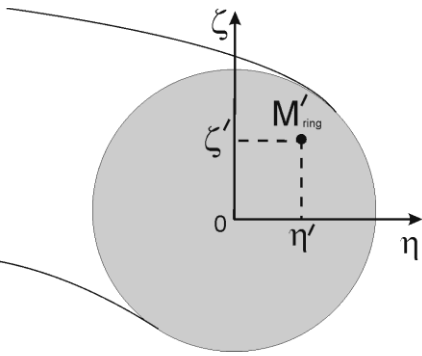
Torus

$R=1$ is major radius of a torus
R_0 is minor radius

Geometrical parameter: $r_0 = R_0/R$



Grav. potential of a homogeneous circular torus



Resulting expression:

$$\boxed{\varphi_{torus}(\rho,\zeta) = \frac{GM}{\pi^2 R \cdot r_0^2} \int_{-r_0}^{r_0} \int_{-\sqrt{r_0^2-\eta'^2}}^{\sqrt{r_0^2-\eta'^2}} \phi'_{ring}(\rho,\zeta;\eta',\zeta')\, d\eta' d\zeta'}$$

where a dimensionless potential of the component ring is:

$$\phi'_{ring}(\eta,\zeta;\eta',\zeta') = \sqrt{\frac{m'(1+\eta')}{\rho}} \cdot K(m')$$

$$m' = \frac{4\rho \cdot (\eta'+1)}{(\rho+\eta'+1)^2 + (\zeta-\zeta')^2}$$

is the parameter of elliptical integral

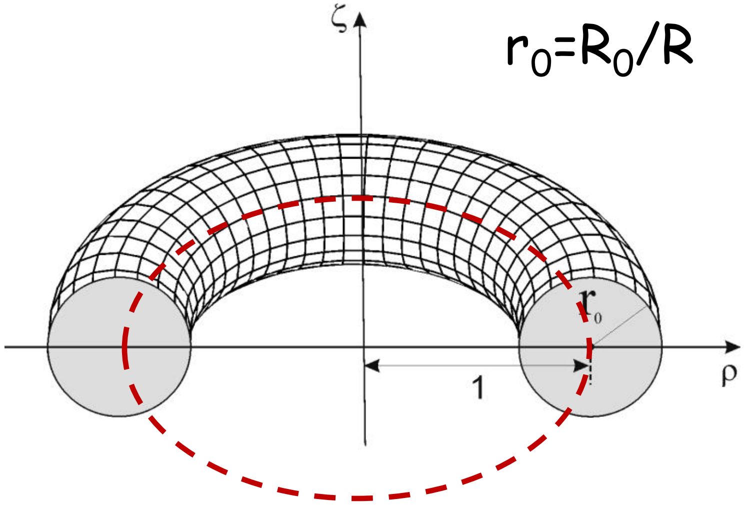
This expression for the torus potential is valid for both the inner and outer points.

Bannikova et al., 2011, MNRAS

Gravitational potential of a homogeneous circular torus

Outer region

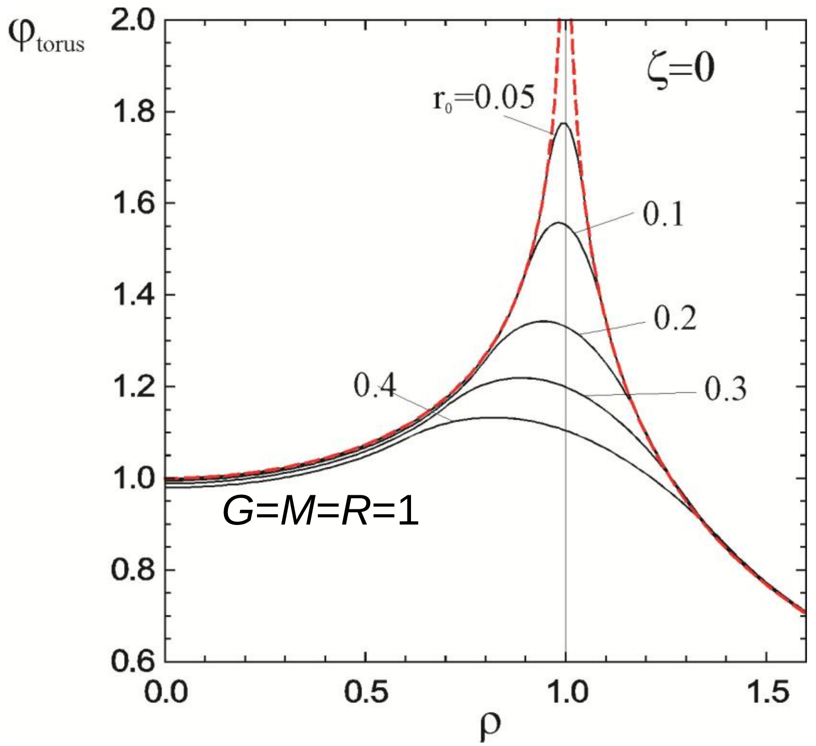
$r_0=R_0/R$



The potential curves for all values of r_0 are seen to be inscribed into the potential curve of an infinitely thin ring of the same mass, located in the torus symmetry plane.

The approximate expression for the torus potential

$$\varphi_{\rm torus}(\rho,\zeta;r_0)\approx\frac{GM}{\pi R}\phi_{\rm c}(\rho,\zeta)\left(1-\frac{r_0^2}{16}+\frac{r_0^2}{16}\frac{\zeta^2-1}{\zeta^2+1}\right)$$

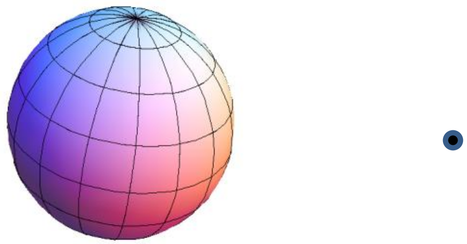


The outer potential of a homogeneous circular torus can be represented with good accuracy by the potential of an infinitely thin ring of the same mass.

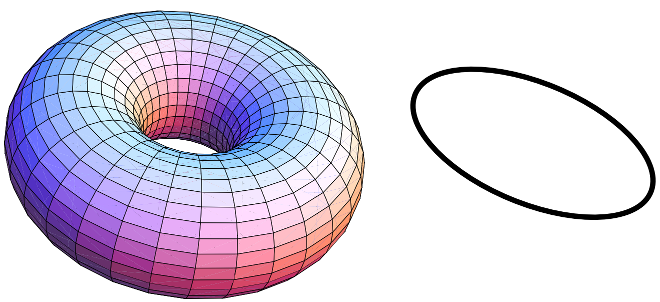
The dependence of the geometrical parameter r_0 appears only in the torus hole

There is analogy with a result for a solid sphere (!)

Outer potential: analogy between results for a solid sphere and a torus



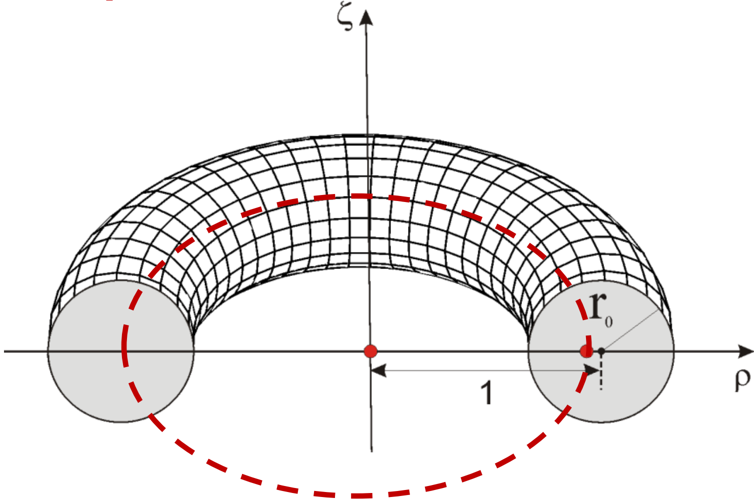
The outer potential of a solid sphere is equal to the potential of a material point



The outer potential of a torus is approximately equal to the potential of a material circle

If we want to investigate the particle motion in the outer region of a torus it is enough to use the potential of a material circle!

Role of a central mass



** Torus is a doubly connected body =>
two weightlessness points in meridional plane:
outer point (in the center of symmetry)
and inner (displace to the center torus)

As a result, the torus must be compressed along the major radius.
To prevent this compression, an orbital motion is necessary:
the gravitational force tending to compress the torus along the major radius is compensated by the centrifugal force.

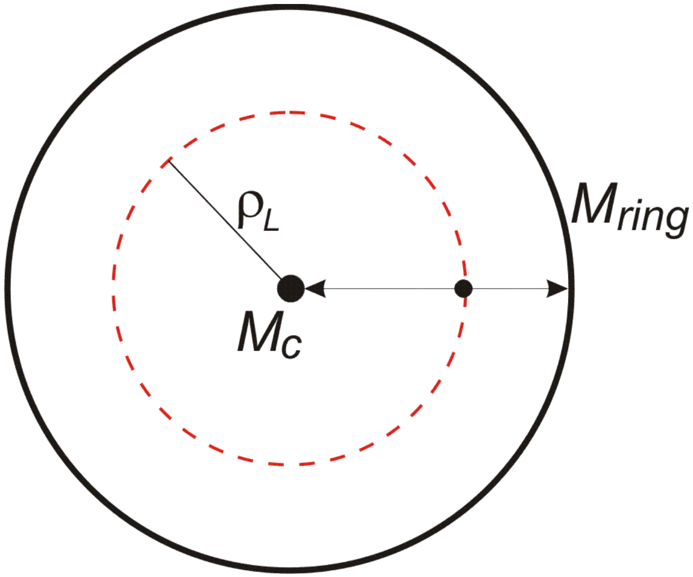
The presence of a central mass is a necessary condition for the stability of a self-gravitating torus.

Lagrangian ring

Let's consider the motion of particle in the grav. field of the thin ring and the central mass in the equatorial plane.

The forces from the thin ring and the central mass must equilibrate at some distance (a region of unstable equilibrium).

We will call the geometrical place of all points, where the balance of these forces are realized, a Lagrangian ring.



$$\boxed{F_{ring,\rho} = F_{c,\rho}}$$

Radial component of the force from the infinitely thin ring

$$F_{ring,\rho} = \frac{GM_{ring}}{\pi R^2}\sqrt{\frac{m}{\rho}}\frac{1}{4\rho(1-m)}\Big[\big(2-m(\rho+1)\big)E(m)-2(1-m)K(m)\Big]$$

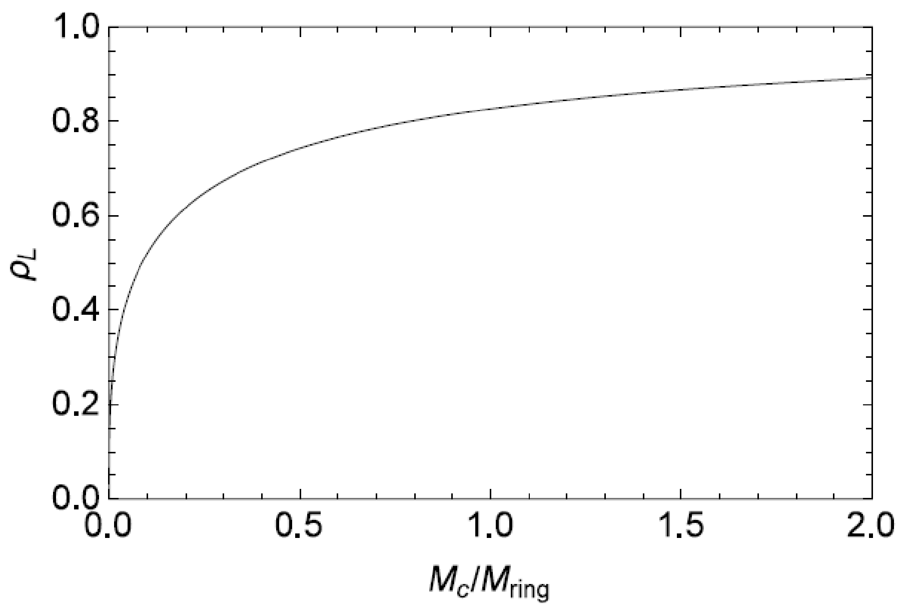
Lagrangian ring

The equation for the radius of Lagrangian ring ($\rho_L<1$)

$$\boxed{\frac{\rho_L}{1-\rho_L}E(\rho_L)-\rho_L K(\rho_L)=q\frac{\pi}{2}}$$

$q=M_c\,/M_{ring}$

K() and E() are the complete elliptical integrals



For $M_c=M_{ring}$ $\rho_L=0.8$

ρ =r/R, R=1 is the radius of the ring

Region of unstable orbits (motion in an equatorial plane)

Effective potential
$$U_{eff} = U_c + U_{ring} + \frac{I_\zeta^2}{2\rho^2}$$

For small values of angular momentum U_{eff} has the minimum which corresponds the stable circle orbits.

Increasing of the moment leads to a shift of the potential minimum in the region of increasing ρ.

Starting from some ρ the effective potential hasn't the minimum and stable orbits don't exist.

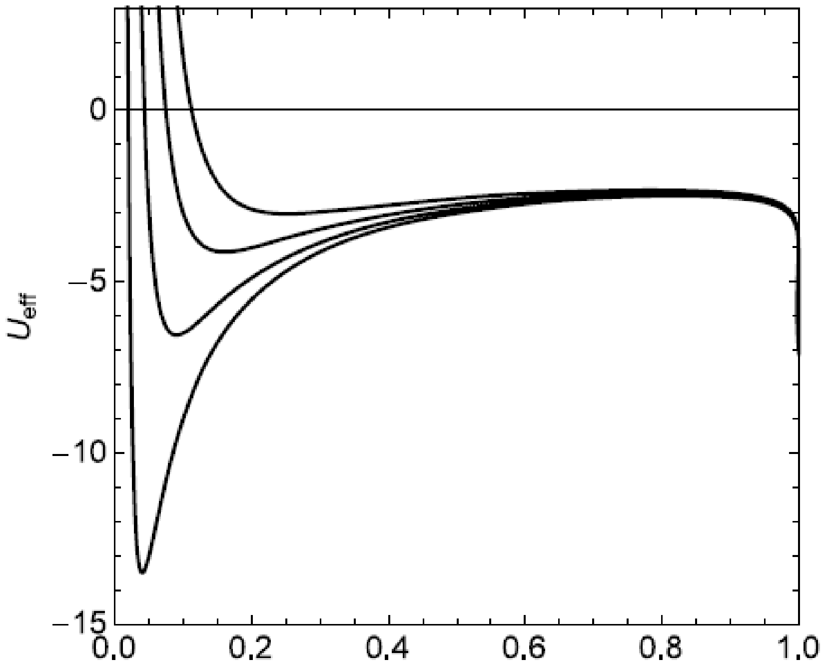


Figure 3. The dependence of effective potential on radial distance for different values of angular moment $I_\zeta = 0.2, 0.3, 0.4, 0.5$.

Region of unstable orbits (motion in an equatorial plane)

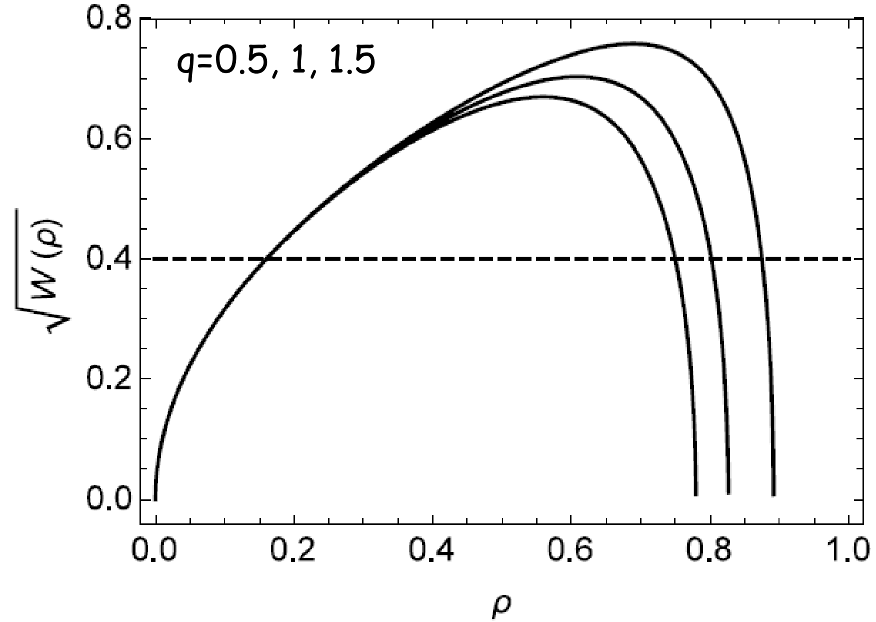
Extremum of U_{eff} leads to the following equation:

$$\frac{I_\zeta^2 R^3}{GM_c} = W(\rho)$$

where

$$W(\rho) = \rho + \frac{2}{\pi} q \rho^2 \left[K(\rho) - \frac{E(\rho)}{1-\rho^2} \right]$$

$q = M_c / M_{ring}$



Graphic solution of this equation shows that there is a restriction on the momentum for which circle orbits exist.

Two solutions for ρ correspond stable and unstable circle orbits.
For $M_c = M_{ring}$ the last stable orbit corresponds to $\rho = 0.6$ and $I_{max} = 0.7$.

Region of unstable orbits (motion in an equatorial plane)

The example of unstable orbit for the different initial values of radius and velocity

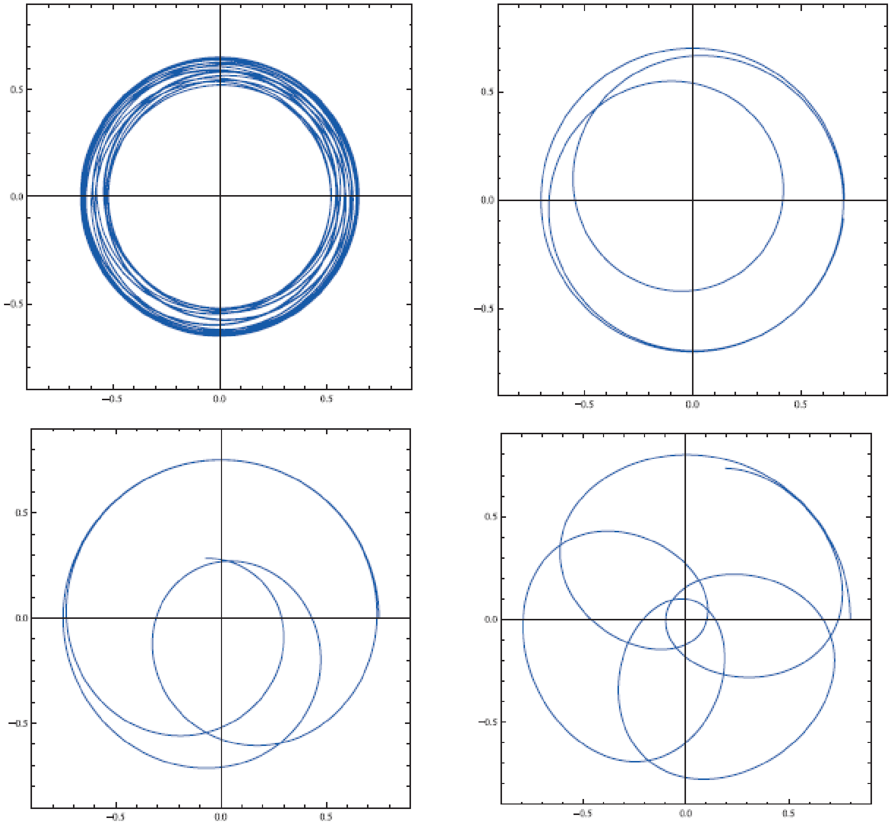


Figure 5. Unstable orbits of a test particle in equatorial plane for the initial conditions: $y_0 = 0$, $V_{x0} = 0$ a) $x_0 = 0.65$, $V_{y0} = 1.072$, b) $x_0 = 0.7$,$V_{y0} = 0.955$, c) $x_0 = 0.75$,$V_{x0} = 0.795$, d) $x_0 = 0.8$, $V_{y0} = 0.513$. For all cases $M_c = G = R = 1$.

Equation of motion

$$\ddot{\vec{\rho}} = \frac{GM_{ring}}{\pi}\frac{\vec{\rho}}{\rho^{5/2}}f(\rho,z) - GM_c\frac{\vec{\rho}}{(\rho^2+z^2)^{3/2}} \qquad (12)$$

$$\ddot{z} = -\frac{GM_{ring}}{\pi}\frac{z}{\rho^{1/2}}\frac{\sqrt{m}}{(\rho-1)^2+z^2}E(m) - GM_c\frac{z}{(\rho^2+z^2)^{3/2}} \qquad (13)$$

where

$$f(\rho,z) = \frac{\sqrt{m}}{4(1-m)}\left[(2-(\rho+1)m)E(m) - 2(1-m)K(m)\right],$$

$\vec{\rho} = (x,y)$ and all coordinates are dimensionless

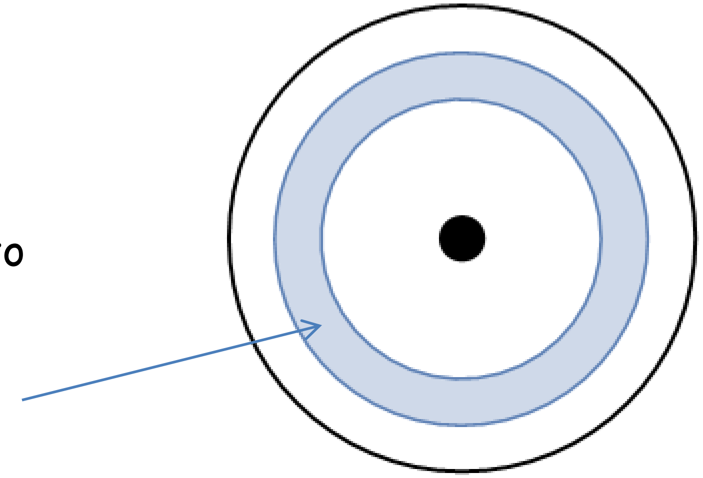
Region of unstable orbits
(motion in an equatorial plane)

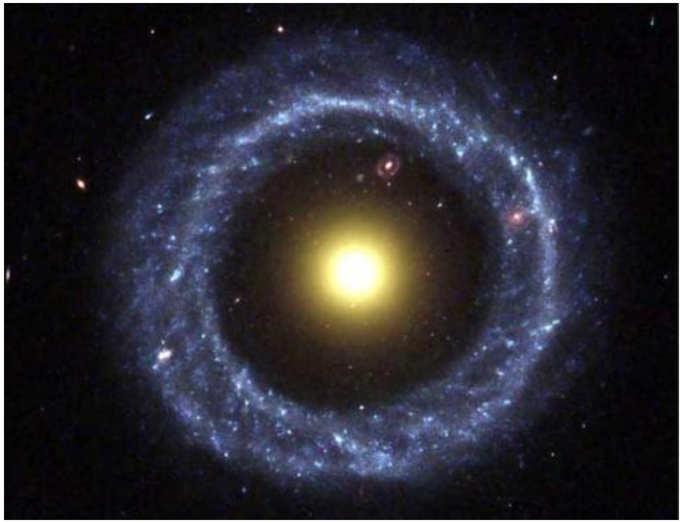
Thus there is a region of unstable orbits bounded by the radius of last stable orbit and Lagrangian ring.

1. Gravity of the central mass dominates near to M_c – it is possible formation of keplerian disk

2. Region of unstable orbits – competition between the grav. forces from M_c and M_{ring}

3. Behind the Lagrangian ring the forces from ring dominate and particle trajectory enwinds a ring.





Frequent collisions are possible in the region of unstable orbits and, as a result, this region could be cleaned off the matter.

It is possible that the gap in Hoag's object arose due to influence of the region of the unstable orbits (?)

Disappearance of Lagrangian and weightlessness rings

Substituting torus by the ring
we simplify the problem about motion
of the particle.

This allow us to see some gravitational
properties of the torus.

LR and WR disappear for more thick
torus r_0>0.17 =>

The homogeneous circular torus
can't exist.

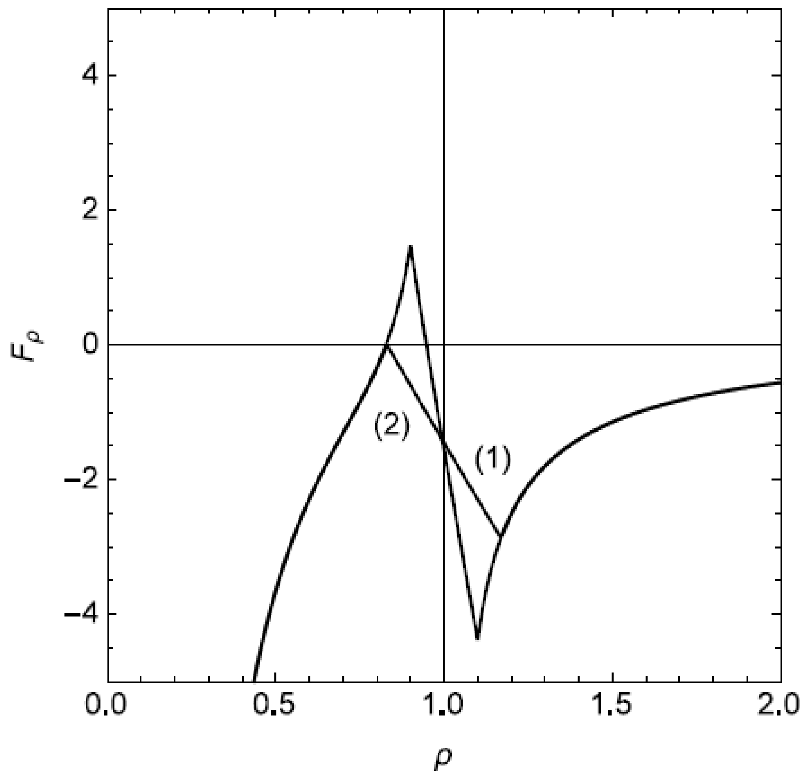


Figure 9. Gravitational radial force from a homogeneous citcular torus and the central mass for two geometrical parameters: (1) $r_0 = 0.1$ and (2) $r_0 = 0.17$.

What is the shape and density of self-gravitating torus?

Self-gravitating torus in the field of the central mass: N-body simulation

- Initial condition: Keplerian torus
- $M_{torus}=(0.02\text{-}0.1)M_c$
- Number of particles $N \approx 10^4$

- Gravitating particles are the Plammer's spheres with radius $\varepsilon=0.01$

- The equations of motion $$\mathbf{a}_i = -\frac{GM_c}{R^2}\frac{\mathbf{r}_i}{r_i^3} + \frac{\mathbf{F}_i}{m_i}$$

The total gravitational force acting on *i*-th particle

$$\mathbf{F}_i = -\frac{Gm_i}{R^2}\sum_{j=1}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{\left(|\mathbf{r}_i - \mathbf{r}_j|^2 + \varepsilon^2\right)^{3/2}}$$

Elliptical Keplerian torus

Colors show the particles which has similar eccentricities.

N=8192

All particles are gathered in one plane.

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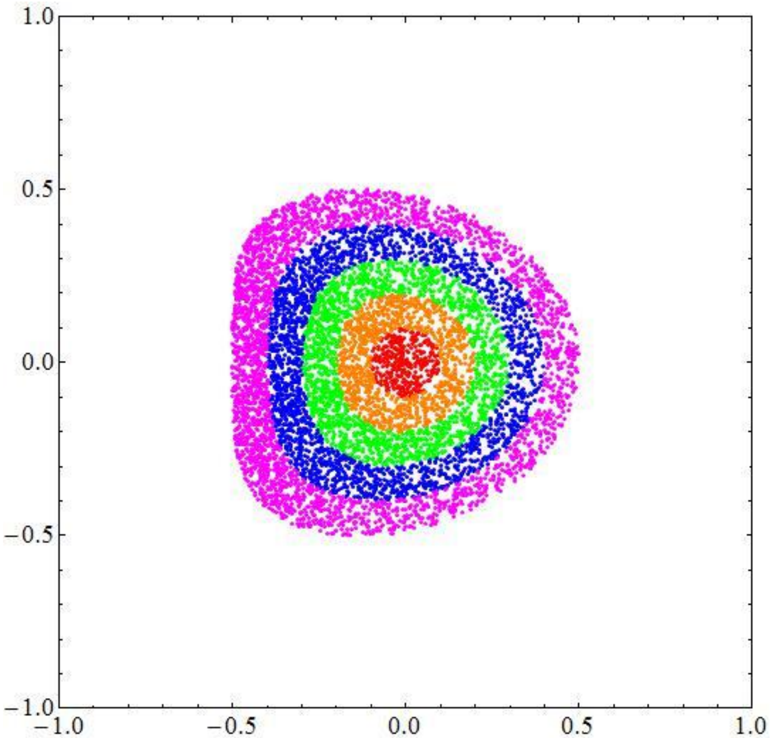
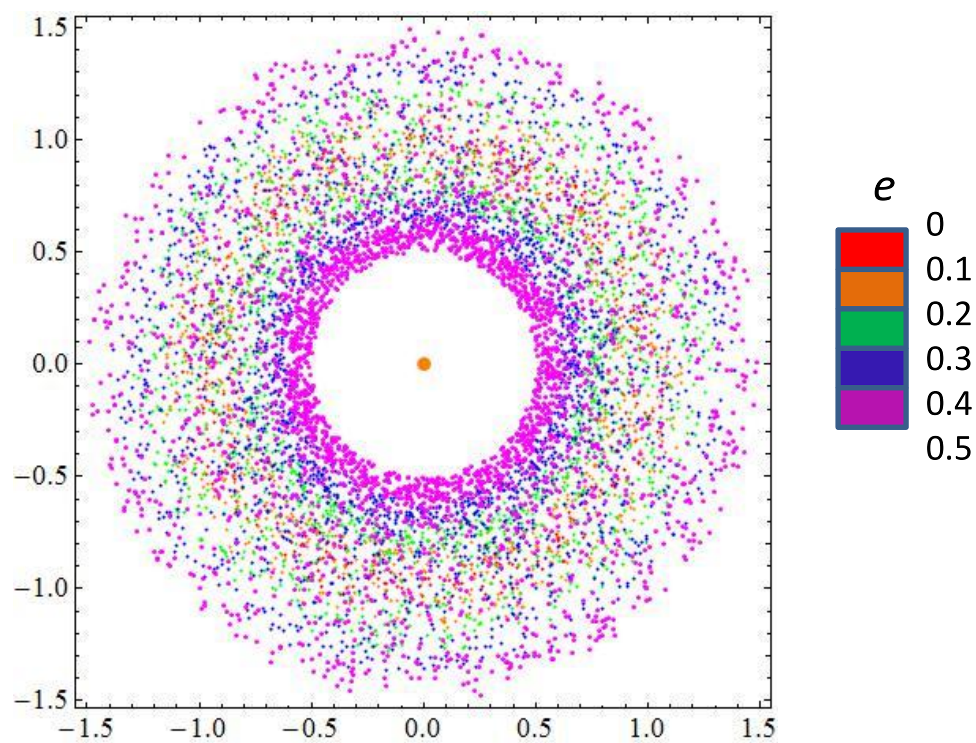
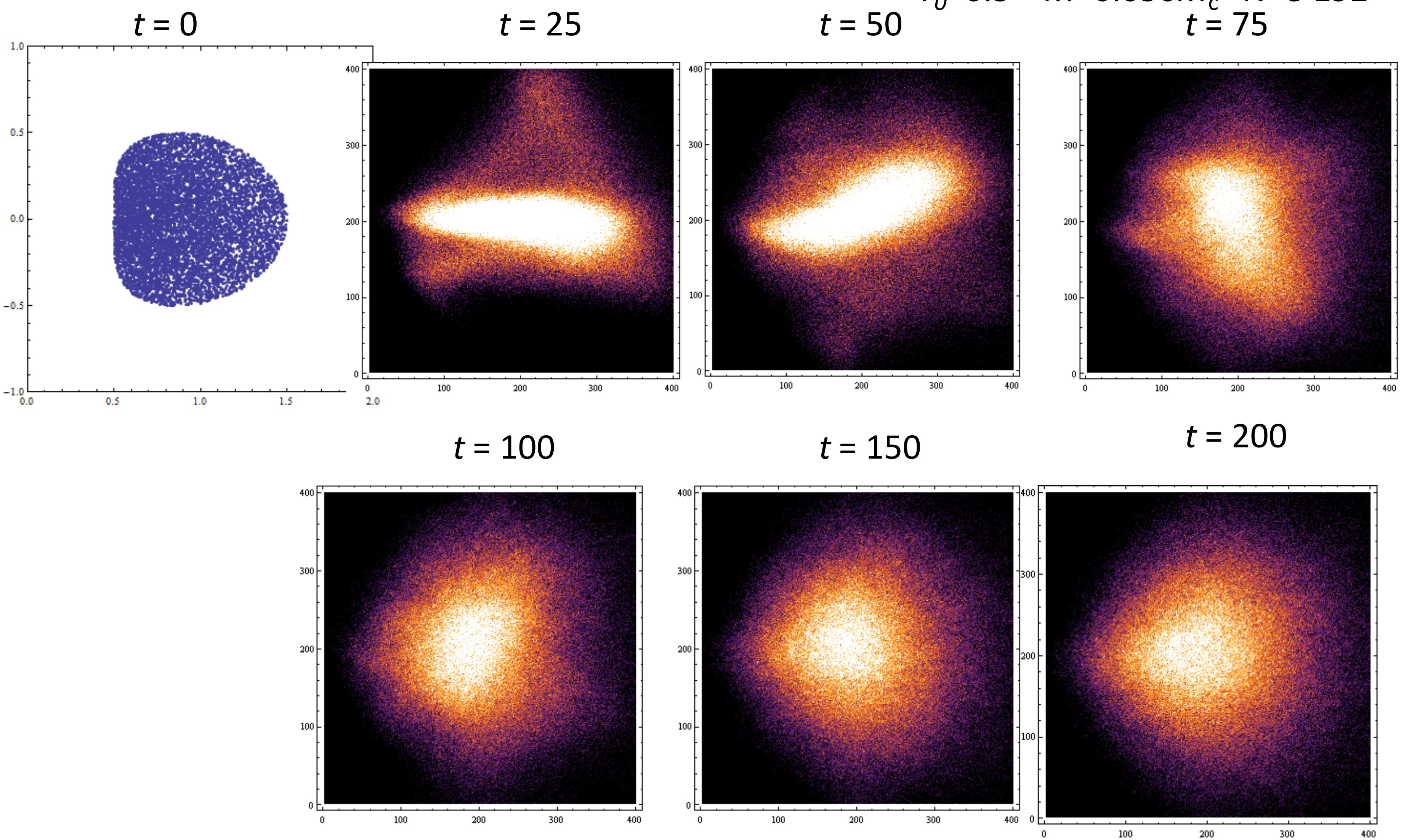




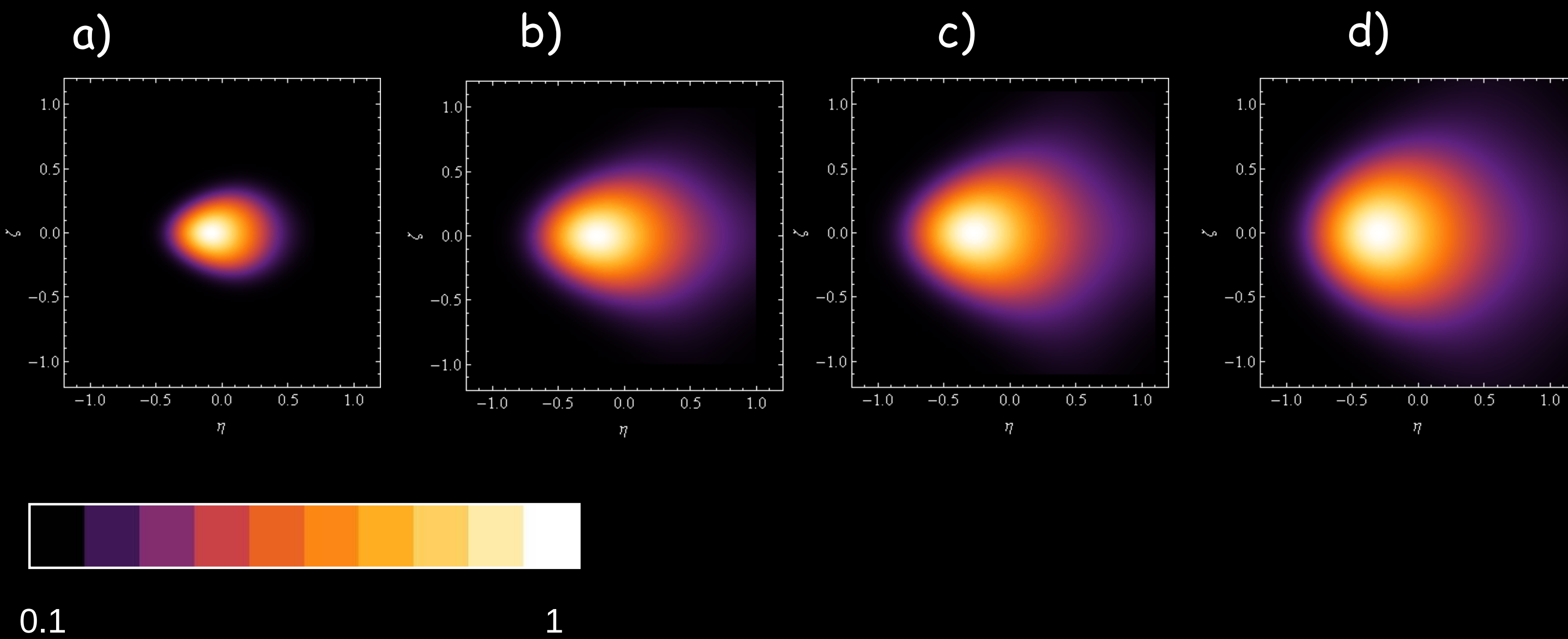
Circle obits are located in the center of cross-section
Elliptical orbits fill in the boundaries of the torus

Initial condition: Keplerian torus

r_0=0.5 M=0.056M_c N=8 192



Density distribution in torus cross-section



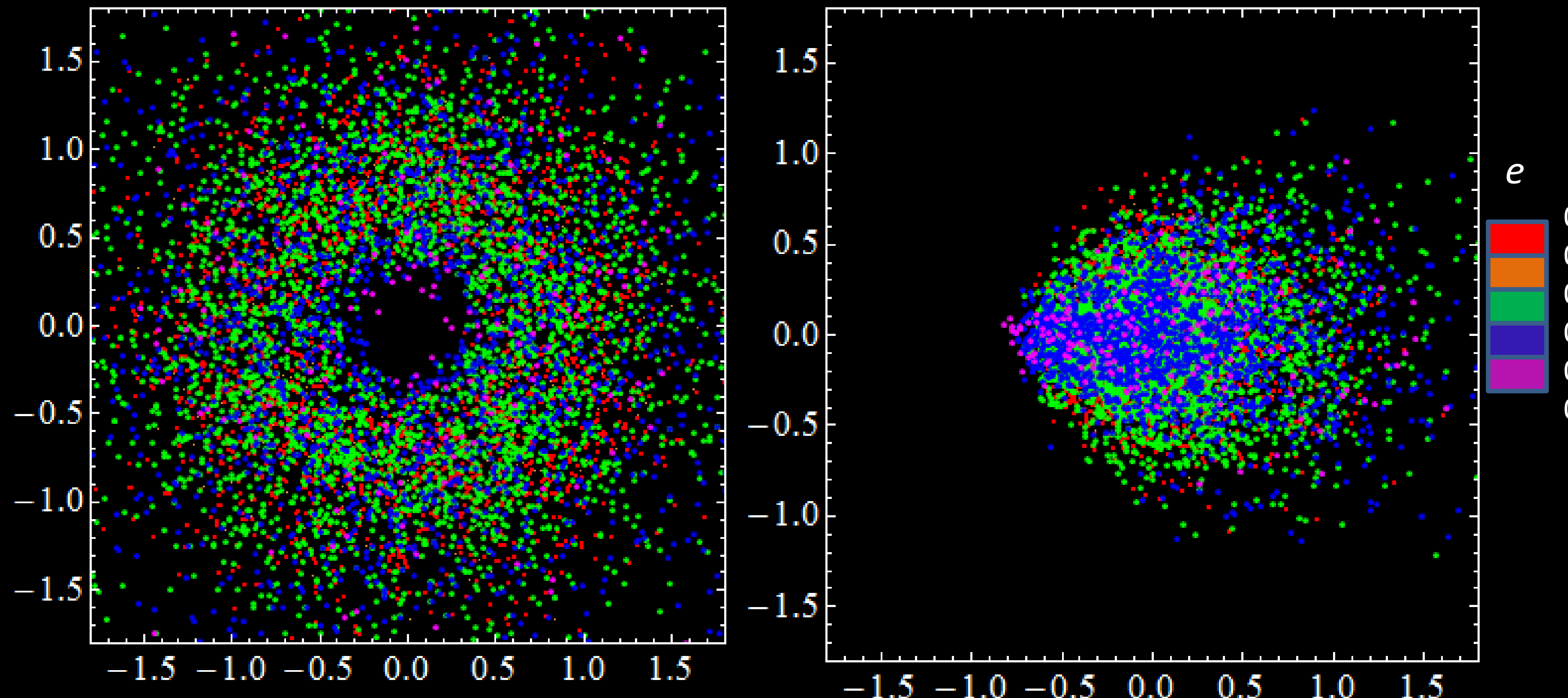
An average density distribution of particles in equilibrium cross-section of torus N=16 384 :
a) r0=0.3, $M_{torus}=0.02M_c$, b) r0=0.5, $M_{torus}=0.056M_c$ c) r0=0.6, $M_{torus}=0.08M_c$, d)r0=0.7, $M_{torus}=0.11M_c$. The parameters are chosen so as in all cases the tori have the same values of initial volume density.

Equilibrium cross-section of self-gravitating torus has an oval shape with Gaussian distribution of particles .

Bannikova et al. 2012, MNRAS

Distribution of particles in the thick torus

$M_{torus}=0.05M_c$ r0=0.5 N=8192 dt=1 000



The inner edge of the torus *is formed by particles* that are moving in elliptical orbits and pass through the pericentre, while the outer edge *is formed by particless that pass through* the apocentre.

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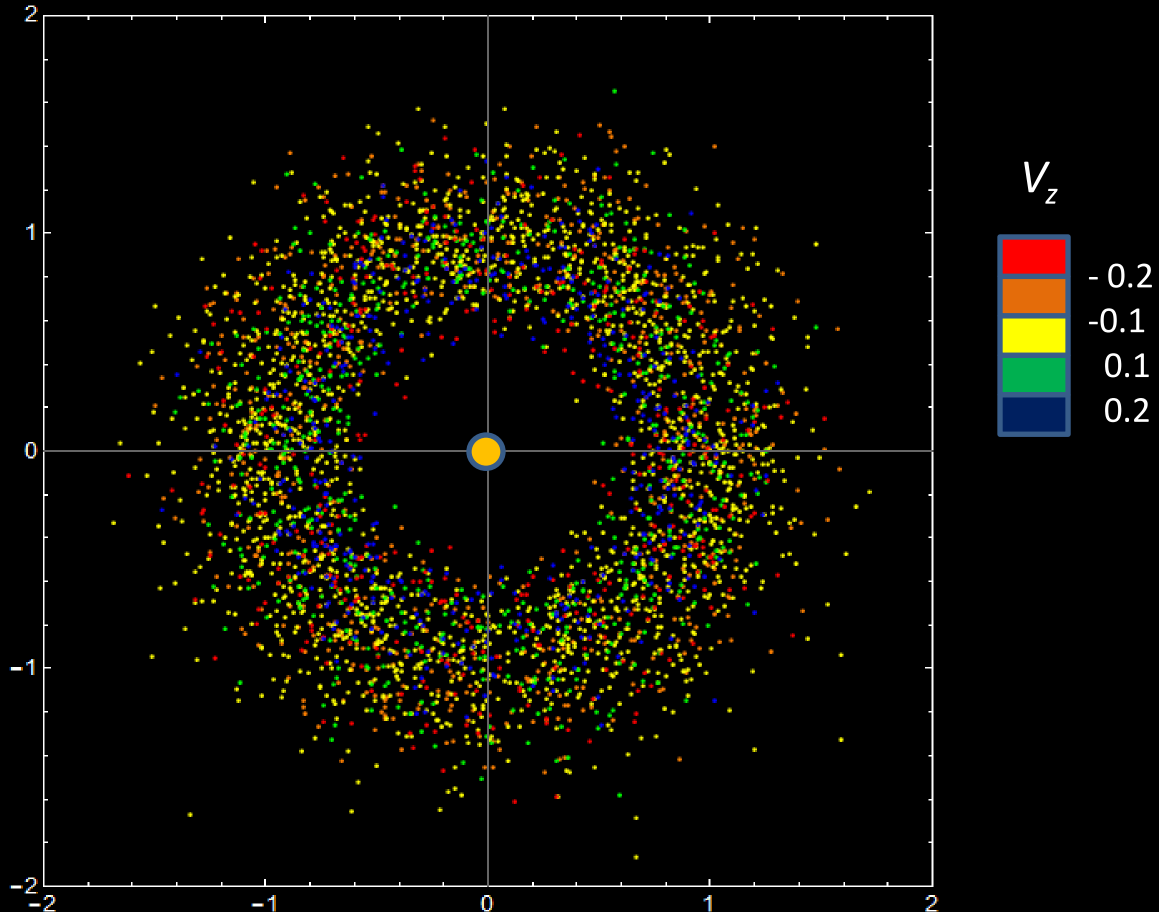
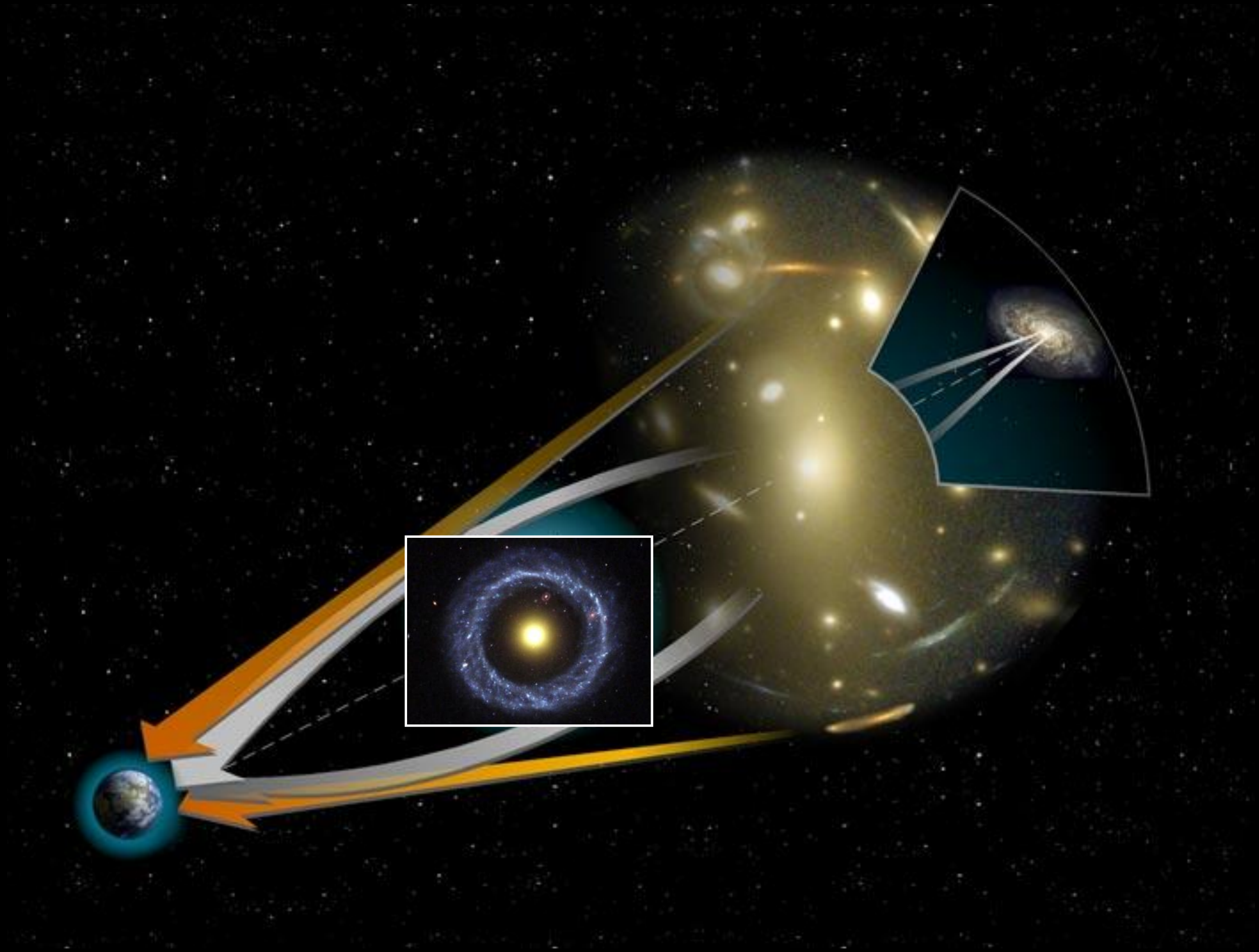
Distribution of z-component of velocity

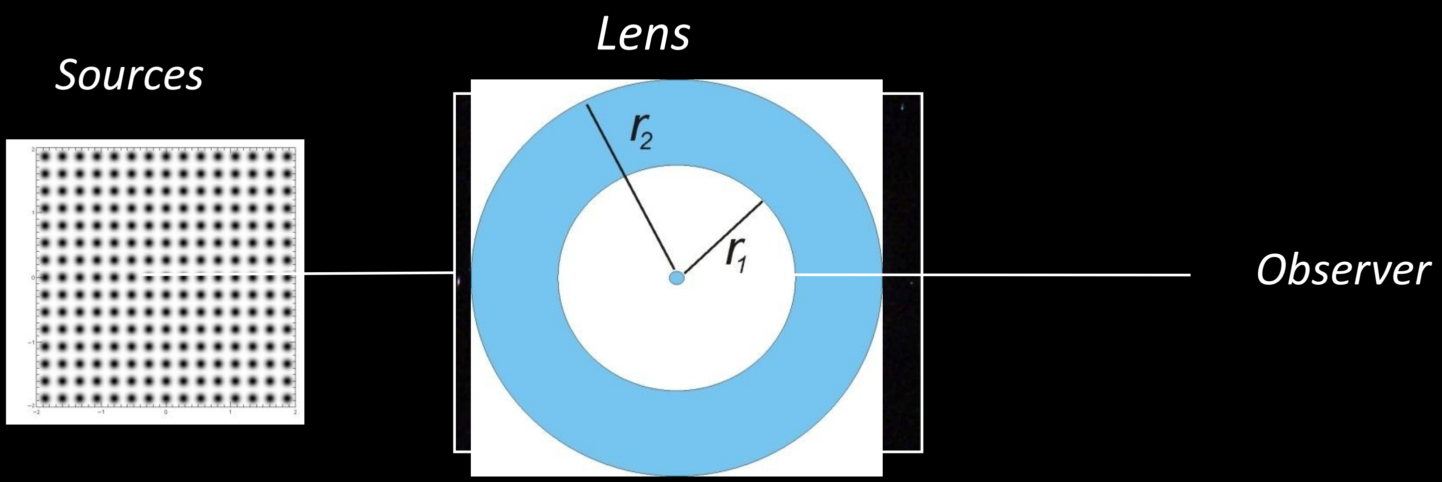
M_{torus}=0.02M_c r0=0.2 N=8192

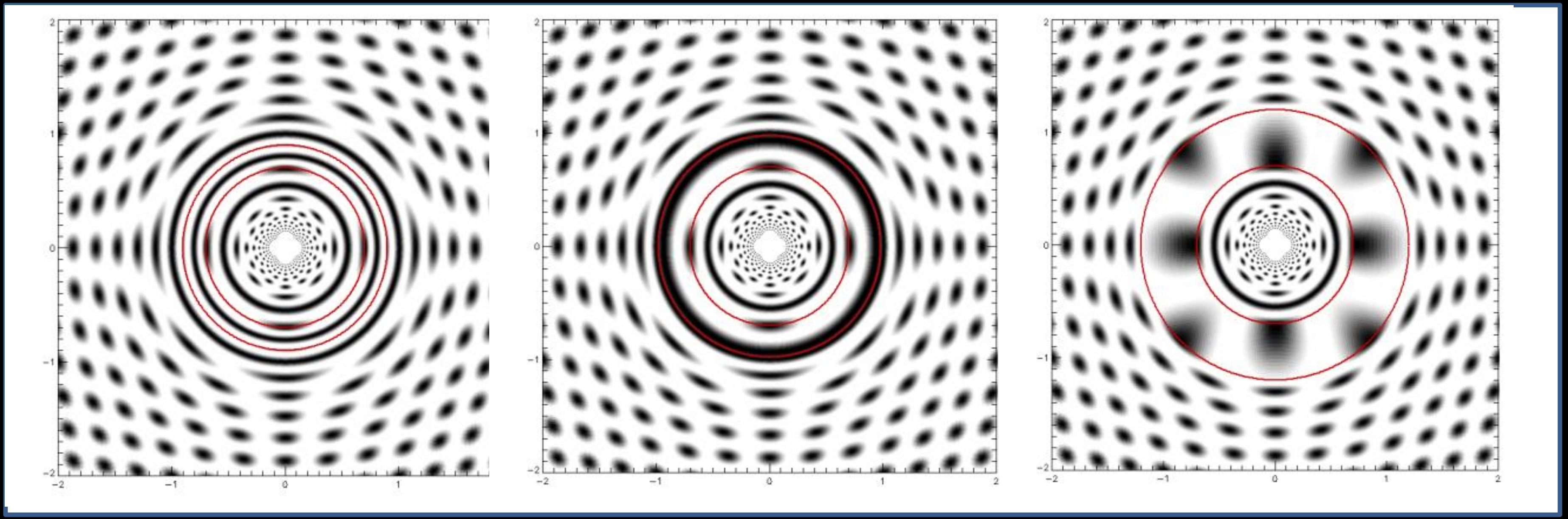


What can we see if a gravitational lens is a ring galaxy?



Three Einstein rings





Bannikova & Kotwitsky, 2014, MNRAS

Conclusion

- The outer gravitational potential of the homogeneous circular torus is approximately equal to the potential of infinitely thin ring.

- There are Lagrangian ring and the region of unstable orbit in the system "infinitely thin ring+Mc".

- The equilibrium cross-section of the self-gravitating torus has an oval shape with Gaussian density distribution.

- Analysis of the Einstein rings arising due to lensing by a disc with the hole and a central mass has shown a diversity of possible cases. In such system one, two and three Einstein rings can form.